Lecture 34

Prop: (Chernoff bound, lower):

For $t<0$, $P(X \leq a) \leq \frac{M_X(t)}{e^{ta}}$.

Pf: Same as other side Chernoff bound: If $t<0$ then the map $x \mapsto e^{tx}$ is monotone decreasing

So
$$P(X \leq a) \underset{\substack{\uparrow \\ \text{since} \\ x \mapsto e^{tx} \text{ is} \\ \text{decreasing.}}}{=} \underbrace{P(e^{tX} \geq e^{ta}) \leq \frac{M_X(t)}{e^{ta}}}_{\text{by Markov's inequality.}}$$

$\square$.

Prop: (Chebyshev's Inequality).

Let X be a RV with finite mean μ and variance σ^2.

Then, for any k
$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}.$$

Pf: Since $(X-\mu)^2$ is non-negative, Markov's Inequality gives
$$P((X-\mu)^2 \geq k^2) \leq \frac{E((X-\mu)^2)}{k^2} = \frac{\sigma^2}{k^2}.$$

But
$$P((X-\mu)^2 \geq k^2) = P(|X-\mu| \geq k),$$ giving the result.

- For the rest of the course we will work toward proving the central limit theorem, which is one of the central results in probability/stats.

<u>Defn</u>: Let $X_1, X_2, X_3, \ldots,$ be a sequence of random variables. Let $F_n(t)$ be the cdf of X_n, and let X be some distribution with cdf $F(t)$. Then X_n "converges in distribution" to X iff

$$\lim_{n \to \infty} F_n(t) = F(t)$$

for all t for which $F(t)$ is continuous.

<u>Idea</u>: "Convergence in distribution" roughly means that the X is a better and better approximation of X_n as n gets large.

<u>Theorem (Central Limit Theorem)</u>

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed RV's with mean μ and variance σ^2. Then

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

converges in distribution to $Z = N(0,1)$.

For all intents and purposes, the theorem says that for large n, the sample mean is approximately normal, i.e.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Why? For large n,

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \approx N(0,1).$$

So

$$X_1 + \cdots + X_n - n\mu \approx \underbrace{\sigma\sqrt{n}\, N(0,1)}_{N(0,\,\sigma^2 n)}.$$

So

$$X_1 + \cdots + X_n \approx \underbrace{N(0, \sigma^2 n) + n\mu}_{N(n\mu,\,\sigma^2 n)}.$$

Dividing by n:

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n} = \underbrace{\frac{1}{n} N(n\mu, \sigma^2 n)}_{N\left(\frac{n\mu}{n},\, \frac{\sigma^2 n}{n^2}\right) = N\left(\mu, \frac{\sigma^2}{n}\right)}$$

To prove the theorem, we use the following lemma, which we won't prove:

<u>Lemma:</u> Let $\{Z_n\}_{n\geq 1}$ have mgf's $M_{Z_n}(t)$ and let Z have mgf $M_Z(t)$. Then if

$$\forall t \quad \lim_{n\to\infty} M_{Z_n}(t) = M_Z(t)$$

then $\{Z_n\}$ converges in distribution to Z.